

GROWTH OF CRITICAL POINTS IN ONE-DIMENSIONAL LATTICE SYSTEMS

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ABSTRACT. We study the growth of the numbers of critical points in one-dimensional lattice systems by using (real) algebraic geometry and the theory of homoclinic tangency.

1. INTRODUCTION: ONE-DIMENSIONAL LATTICE SYSTEM

Let M be a compact connected C^∞ manifold without boundary. Let $f : M \times M \rightarrow \mathbb{R}$ be a C^∞ function. For positive integers n , we define $f_n : M^{n+1} \rightarrow \mathbb{R}$ by setting

$$(1) \quad f_n(p_1, p_2, \dots, p_{n+1}) := \sum_{i=1}^n f(p_i, p_{i+1}).$$

Bertelson-Gromov [4] proposed the study of this kind of functions. (See also Bertelson [3].) Let $\text{Cr}(f_n)$ be the set of critical points of f_n . We are interested in the asymptotic behavior of $\text{Cr}(f_n)$ as $n \rightarrow \infty$. (The paper [7] studies an asymptotic behavior of critical values of f_n .)

Naively speaking, the study of f_n is a model of a 1-dimensional crystal (lattice system) which consists of $n + 1$ particles. We assume that the manifold M is the configuration space of each particle and that $f(x, y)$ is the potential function describing the interaction between two adjacent particles. Then f_n is the total potential energy of the system, and the critical points of f_n are its stationary states.

Our view point and methods are motivated by the works of Artin-Mazur [1] and Kaloshin [12, 13]. They studied the growth of periodic points of diffeomorphisms of manifolds by using (real) algebraic geometry and the theory of homoclinic tangency. We develop analogous methods for the study of $\text{Cr}(f_n)$.

Let $C^\infty(M \times M)$ be the space of real valued C^∞ functions in $M \times M$ with the C^∞ topology. Our first main result is the following theorem. (Recall that a smooth function on a manifold is called a Morse function if all its critical points are non-degenerate.)

Theorem 1.1. *There exists a dense subset $\mathcal{D} \subset C^\infty(M \times M)$ such that every $f \in \mathcal{D}$ satisfies the following two conditions:*

(i) *For all positive integers n , the functions $f_n : M^{n+1} \rightarrow \mathbb{R}$ are Morse functions.*

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(ii) *There exists a positive real number d (which depends on f) such that for all $n \geq 1$*

$$\#\mathrm{Cr}(f_n) \leq d^n.$$

Here $\#\mathrm{Cr}(f_n)$ is the number of the critical points of f_n .

From the condition (i) in Theorem 1.1 and the Morse inequality ([17]), we have

$$\#\mathrm{Cr}(f_n) \geq \sum_{k \geq 0} \dim(H^k(M^{n+1}; \mathbb{Z}_2)) = \left(\sum_{k \geq 0} \dim H^k(M; \mathbb{Z}_2) \right)^{n+1}$$

for all $f \in \mathcal{D}$. Hence, from the condition (ii),

$$\left(\sum_{k \geq 0} \dim H^k(M; \mathbb{Z}_2) \right)^{n+1} \leq \#\mathrm{Cr}(f_n) \leq d^n \quad (f \in \mathcal{D}, n \geq 1).$$

Therefore, if $\dim M \geq 1$, $\#\mathrm{Cr}(f_n)$ has an exponential growth for every $f \in \mathcal{D}$.

Remark 1.2. For each $n \geq 1$, the condition that f_n is a Morse function is an open condition for $f \in C^\infty(M \times M)$. Therefore Theorem 1.1 (condition (i)) implies that the set

$$\{f \in C^\infty(M \times M) \mid \text{All } f_n \ (n \geq 1) \text{ are Morse functions}\}$$

is a residual subset of $C^\infty(M \times M)$. (Recall that a subset of a topological space is said to be residual if it contains a countable intersection of open dense subsets.) This fact was already proved in Asaoka-Fukaya-Tsukamoto [2, Theorem 1.2]. The argument in the present paper is totally different from that in [2]. The argument in [2] is much more elementary. It uses only elementary results in differential topology. On the other hand, the argument of the present paper uses two very big theorems: Nash-Tognoli-King's theorem in real algebraic geometry ([21], [23], [15]) and Hironaka's resolution of singularities ([11]). The important point is that we can achieve the condition (ii) in Theorem 1.1. This is the new point of the present paper.

By the above remark, the sets $\mathrm{Cr}(f_n)$ are finite sets for generic (i.e. residual) $f : M \times M \rightarrow \mathbb{R}$. Theorem 1.1 shows a regular behavior of $\mathrm{Cr}(f_n)$ for “many” (i.e. dense) f . Next we will show that there exists a very wild phenomenon. We concentrate on the case $M = S^1 := \mathbb{R}/2\pi\mathbb{Z}$.

Theorem 1.3. *There exists a non-empty open subset \mathcal{U} of $C^\infty(S^1 \times S^1)$ such that the set*

$$\left\{ f \in \mathcal{U} \mid \limsup_{n \rightarrow \infty} \frac{\#\mathrm{Cr}(f_n)}{a_n} \geq 1 \right\}$$

is residual in \mathcal{U} for any given sequence $(a_n)_{n \geq 1}$ of positive integers.

For example, this implies that the set

$$\left\{ f \in \mathcal{U} \mid \limsup_{n \rightarrow \infty} \frac{\#\mathrm{Cr}(f_n)}{\exp(\exp(n))} \geq 1 \right\}$$

is residual in the above given open set \mathcal{U} . This shows a drastically unstable behavior of $\text{Cr}(f_n)$ over $f \in \mathcal{U}$. (Note that $\#\text{Cr}(f_n)$ has an exponential growth on a dense subset of \mathcal{U} .)

It is interesting to see that there also exists a “stable region”:

Theorem 1.4. *There exist $d > 0$ and a non-empty open set $\mathcal{V} \subset \mathcal{C}^\infty(S^1 \times S^1)$ such that every $f \in \mathcal{V}$ satisfies the following conditions:*

- (i) *All $f_n : (S^1)^{n+1} \rightarrow \mathbb{R}$ ($n \geq 1$) are Morse functions.*
- (ii) *For all $n \geq 1$, $\#\text{Cr}(f_n) = d^{n+1}$.*

The following question seems interesting.

Problem 1.5. Find a characterization of a function $f : M \times M \rightarrow \mathbb{R}$ which admits a neighborhood $\mathcal{W} \subset C^\infty(M \times M)$ such that for every $g \in \mathcal{W}$ all functions $g_n : M^{n+1} \rightarrow \mathbb{R}$ ($n \geq 1$) are Morse functions.

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2. PROOF OF THEOREM 1.1

The essential ingredient of the proof of Theorem 1.1 is algebraic geometry. As far as the authors know, the idea to use Nash’s theorem and algebraic geometry in the study of C^∞ manifolds goes back to Artin-Mazur [1]. (For recent results on the Artin-Mazur type problem, see Kaloshin [12, 13].)

2.1. Preliminary. Let M be a compact connected C^∞ manifold without boundary. The next proposition will be proved later (Section 2.3).

Proposition 2.1. *There exist homogeneous polynomials (of real coefficients) $F_i(X_0, X_1, \dots, X_d)$ ($1 \leq i \leq R$) in $\mathbb{R}[X_0, X_1, \dots, X_d]$ satisfying the following three conditions.*

- (i) *The scheme $\text{Proj}(\mathbb{C}[X_0, X_1, \dots, X_d]/(F_1, \dots, F_R))$ is an equidimensional regular scheme. (Set $\dim \text{Proj}(\mathbb{C}[X_0, X_1, \dots, X_d]/(F_1, \dots, F_R)) = d - r$.) This implies the following:*

Set $X := \{[X_0 : \dots : X_d] \in \mathbb{P}^d(\mathbb{C}) \mid F_i(X_0, \dots, X_d) = 0 \ (1 \leq i \leq R)\}$. Then X is a complex submanifold of $\mathbb{P}^d(\mathbb{C})$, and the (complex) dimension of every connected component of X is equal to $d - r$. Moreover

$$\text{rank} \begin{bmatrix} \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_d} \\ \frac{\partial F_2}{\partial X_0} & \frac{\partial F_2}{\partial X_1} & \cdots & \frac{\partial F_2}{\partial X_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_0} & \frac{\partial F_R}{\partial X_1} & \cdots & \frac{\partial F_R}{\partial X_d} \end{bmatrix} = r \quad \text{on } X.$$

(ii) X transversally intersects with the hyperplane $\{X_0 = 0\}$ in $\mathbb{P}^d(\mathbb{C})$. This implies the following: Set $X_\infty := X \cap \{X_0 = 0\}$. Then

$$\text{rank} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_d} \\ \frac{\partial F_2}{\partial X_0} & \frac{\partial F_2}{\partial X_1} & \cdots & \frac{\partial F_2}{\partial X_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_0} & \frac{\partial F_R}{\partial X_1} & \cdots & \frac{\partial F_R}{\partial X_d} \end{bmatrix} = 1 + \text{rank} \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_d} \\ \frac{\partial F_2}{\partial X_1} & \cdots & \frac{\partial F_2}{\partial X_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_1} & \cdots & \frac{\partial F_R}{\partial X_d} \end{bmatrix} = 1 + r \quad \text{on } X_\infty.$$

(iii) Let $\mathbb{R}^d := \{[1 : x_1 : x_2 : \cdots : x_d] \in \mathbb{P}^d(\mathbb{C}) \mid x_1, \dots, x_d \in \mathbb{R}\}$. Then $\underline{X}^\mathbb{R} := X \cap \mathbb{R}^d$ is diffeomorphic to M . Here $\underline{X}^\mathbb{R}$ becomes a C^∞ submanifold of \mathbb{R}^d by the above condition (i). (We have $\dim_\mathbb{R} \underline{X}^\mathbb{R} = d - r$.) We fix a diffeomorphism between M and $\underline{X}^\mathbb{R}$ and identify them.

Let $\mathbb{C}^d := \{[1 : x_1 : \cdots : x_d] \in \mathbb{P}^d(\mathbb{C}) \mid x_1, \dots, x_d \in \mathbb{C}\}$, and set $\underline{X} := X \cap \mathbb{C}^d$. \underline{X} is a complex submanifold of \mathbb{C}^d by the condition (i) in Proposition 2.1.

Example 2.2. If $M = S^1$, then $R = 1$ and $F_1(X_0, X_1, X_2) = -X_0^2 + X_1^2 + X_2^2$ satisfy the conditions of Proposition 2.1. In this case, we have $d = 2$ and $r = 1$.

Lemma 2.3. For any positive integer N , there is a homogeneous polynomial $\Psi(X_0, X_1, \dots, X_d) \in \mathbb{C}[X_0, X_1, \dots, X_d]$ of degree N satisfying the following two conditions.

(i)

$$\text{rank} \begin{bmatrix} \frac{\partial \Psi}{\partial X_1} & \cdots & \frac{\partial \Psi}{\partial X_d} \\ \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_1} & \cdots & \frac{\partial F_R}{\partial X_d} \end{bmatrix} = r + 1 \quad \text{on } X_\infty = X \cap \{X_0 = 0\}.$$

(ii) The holomorphic function $\psi : \underline{X} \rightarrow \mathbb{C}$, $[1 : x_1 : \cdots : x_d] \mapsto \Psi(1, x_1, \dots, x_d)$, is a Morse function, i.e., the Hessians of ψ at the critical points are regular.

Proof. From the conditions (i) and (ii) in Proposition 2.1, X_∞ is a complex submanifold of $\{X_0 = 0\} = \mathbb{P}^{d-1}(\mathbb{C})$ of codimension r . For any $N \geq 1$, we can choose a homogeneous polynomial $\Psi_0(X_1, \dots, X_d) \in \mathbb{C}[X_1, \dots, X_d]$ of degree N such that the hypersurface $\{\Psi_0 = 0\}$ is non-singular ($\text{grad} \Psi_0 \neq 0$ on $\{\Psi_0 = 0\}$) and transversally intersects with X_∞ in $\mathbb{P}^{d-1}(\mathbb{C})$ (the N -times Segre embeddings and Bertini's theorem). This implies

$$\text{rank} \begin{bmatrix} \frac{\partial \Psi_0}{\partial X_1} & \cdots & \frac{\partial \Psi_0}{\partial X_d} \\ \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_1} & \cdots & \frac{\partial F_R}{\partial X_d} \end{bmatrix} = r + 1 \quad \text{on } X_\infty.$$

For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{C}^d$, we set $\Psi_{\mathbf{a}}(X_0, X_1, \dots, X_d) := \Psi_0(X_1, \dots, X_d) + X_0^{N-1}(a_1 X_1 + \cdots + a_d X_d)$. The above condition (i) (of this lemma) is an open condition. Hence if we

choose $\mathbf{a} \in \mathbb{C}^d$ sufficiently small in the Euclidean norm, then $\Psi_{\mathbf{a}}$ also satisfies the condition (i). Set $\psi_{\mathbf{a}}([1 : x_1 : \cdots : x_d]) := \Psi_{\mathbf{a}}(1, x_1, \dots, x_d) = \Psi_0(1, x_1, \dots, x_d) + a_1 x_1 + \cdots + a_d x_d$. Let $\mathcal{X} \subset \underline{X} \times \mathbb{C}^d$ be the set of points $([1 : x_1 : \cdots : x_d], \mathbf{a}) \in \underline{X} \times \mathbb{C}^d$ such that $d(\psi_{\mathbf{a}}|_{\underline{X}}) = 0$ at $[1 : x_1 : \cdots : x_d]$. \mathcal{X} is a complex submanifold of $\underline{X} \times \mathbb{C}^d$. By Sard's theorem, we can choose a sufficiently small $\mathbf{a} \in \mathbb{C}^d$ such that \mathbf{a} is a regular value of the projection $\mathcal{X} \rightarrow \mathbb{C}^d$. Thus we can choose a sufficiently small $\mathbf{a} \in \mathbb{C}^d$ such that $\Psi_{\mathbf{a}}$ satisfies the conditions (i) and (ii) of this lemma. \square

2.2. Proof of Theorem 1.1. Let N be a positive integer, and let $V_N \subset \mathbb{C}[x_1, \dots, x_d, y_1, \dots, y_d]$ be the set of polynomials $\varphi(x_1, \dots, x_d, y_1, \dots, y_d)$ satisfying $\deg \varphi \leq N$. Take $\varphi \in V_N$. We define a homogeneous polynomial $\Phi(Z, X_1, \dots, X_d, Y_1, \dots, Y_d) \in \mathbb{C}[Z, X_1, \dots, X_d, Y_1, \dots, Y_d]$ by setting

$$(2) \quad \Phi(Z, X_1, \dots, X_d, Y_1, \dots, Y_d) := Z^N \varphi\left(\frac{X_1}{Z}, \dots, \frac{X_d}{Z}, \frac{Y_1}{Z}, \dots, \frac{Y_d}{Z}\right).$$

For positive integers n , we set

$$(3) \quad \varphi_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}) := \sum_{k=1}^n \varphi(\mathbf{x}_k, \mathbf{x}_{k+1}), \quad \Phi_n(Z, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n+1}) := \sum_{k=1}^n \Phi(Z, \mathbf{X}_k, \mathbf{X}_{k+1})$$

where $\mathbf{x}_k = (x_{k1}, x_{k2}, \dots, x_{kd})$ and $\mathbf{X}_k = (X_{k1}, X_{k2}, \dots, X_{kd})$.

We define $\rho_{n,k}(\varphi)(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1})$ ($1 \leq k \leq n+1$) as the rank of the following matrix:

$$(4) \quad \begin{bmatrix} \frac{\partial \Phi_n}{\partial X_{k1}}(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1}) & \frac{\partial \Phi_n}{\partial X_{k2}}(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1}) & \cdots & \frac{\partial \Phi_n}{\partial X_{kd}}(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \\ \frac{\partial F_1}{\partial X_1}(Z, \mathbf{X}_k) & \frac{\partial F_1}{\partial X_2}(Z, \mathbf{X}_k) & \cdots & \frac{\partial F_1}{\partial X_d}(Z, \mathbf{X}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_R}{\partial X_1}(Z, \mathbf{X}_k) & \frac{\partial F_R}{\partial X_2}(Z, \mathbf{X}_k) & \cdots & \frac{\partial F_R}{\partial X_d}(Z, \mathbf{X}_k) \end{bmatrix}.$$

Consider the following condition for $[Z : \mathbf{X}_1 : \mathbf{X}_2 : \cdots : \mathbf{X}_{n+1}] \in \mathbb{P}^{d(n+1)}(\mathbb{C})$:

$$(5) \quad F_i(Z, \mathbf{X}_k) = 0, \quad \rho_{n,k}(\varphi)(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \leq r \quad (1 \leq i \leq R, 1 \leq k \leq n+1).$$

A point $([1 : \mathbf{x}_1], [1 : \mathbf{x}_2], \dots, [1 : \mathbf{x}_{n+1}]) \in \underline{X}^{n+1}$ is a critical point of the function

$$(6) \quad \varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}, \quad ([1 : \mathbf{x}_1], \dots, [1 : \mathbf{x}_{n+1}]) \mapsto \varphi_n(\mathbf{x}_1, \dots, \mathbf{x}_{n+1})$$

if and only if the point $[1 : \mathbf{x}_1 : \mathbf{x}_2 : \cdots : \mathbf{x}_{n+1}] \in \mathbb{P}^{d(n+1)}(\mathbb{C})$ satisfies the above condition (5).

Lemma 2.4. *There exists $\varphi \in V_N$ satisfying the following (i) and (ii):*

- (i) *For any positive integer n , if a point $[Z : \mathbf{X}_1 : \mathbf{X}_2 : \cdots : \mathbf{X}_{n+1}] \in \mathbb{P}^{d(n+1)}(\mathbb{C})$ satisfies (5), then $Z \neq 0$.*
- (ii) *For all positive integers n , the functions $\varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}$ in (6) are Morse functions.*

Proof. Let $\Psi(X_0, X_1, \dots, X_d)$ be the homogeneous polynomial of degree N given by Lemma 2.3. We set $\varphi(\mathbf{x}, \mathbf{y}) := \Psi(1, \mathbf{x}) + \Psi(1, \mathbf{y})$ ($\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d)$). The polynomials φ_n and Φ_n (defined in (3)) become

$$\varphi_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}) = \Psi(1, \mathbf{x}_1) + 2(\Psi(1, \mathbf{x}_2) + \dots + \Psi(1, \mathbf{x}_n)) + \Psi(1, \mathbf{x}_{n+1}),$$

$$\Phi_n(Z, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n+1}) = \Psi(Z, \mathbf{X}_1) + 2(\Psi(Z, \mathbf{X}_2) + \dots + \Psi(Z, \mathbf{X}_n)) + \Psi(Z, \mathbf{X}_{n+1}).$$

Then the above conditions (i) and (ii) immediately follow from the conditions (i) and (ii) of Lemma 2.3. \square

For $n \geq 1$, we define $V_{N,n} \subset V_N$ as the set of $\varphi \in V_N$ satisfying the following: If a point $[Z : \mathbf{X}_1 : \mathbf{X}_2 : \dots : \mathbf{X}_{n+1}] \in \mathbb{P}^{d(n+1)}(\mathbb{C})$ satisfies (5), then $Z \neq 0$.

Lemma 2.5. *The set $V_{N,n}$ is a non-empty Zariski open subset of V_N . Here we naturally identify V_N with the affine space $\mathbb{C}^{\binom{N+2d}{2d}}$. (Here and in the following in this section, we use only algebraic (not analytic) Zariski open/closed subsets.)*

Proof. From Lemma 2.4, $V_{N,n} \neq \emptyset$. We want to show that this is Zariski open. We define $A \subset V_N \times \mathbb{P}^{d(n+1)-1}(\mathbb{C})$ by setting

$$A := \{(\varphi, [\mathbf{X}_1 : \dots : \mathbf{X}_{n+1}]) \in V_N \times \mathbb{P}^{d(n+1)-1}(\mathbb{C}) \mid \varphi \text{ and } [0 : \mathbf{X}_1 : \dots : \mathbf{X}_{n+1}] \text{ satisfy (5)}\}.$$

A is a Zariski closed subset of $V_N \times \mathbb{P}^{d(n+1)-1}(\mathbb{C})$. Let $\pi : V_N \times \mathbb{P}^{d(n+1)-1}(\mathbb{C}) \rightarrow V_N$ be the natural projection. (Here we consider π as a map in the algebraic category (not in the analytic category).) Since $\mathbb{P}^{d(n+1)-1}(\mathbb{C})$ is complete (see Mumford [20, p. 55, Theorem 1]), $\pi(A)$ is Zariski closed in V_N . Therefore $V_{N,n} = V_N \setminus \pi(A)$ is Zariski open. \square

For $n \geq 1$, we define $U_{N,n} \subset V_{N,n}$ by

$$U_{N,n} := \{\varphi \in V_{N,n} \mid \varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C} \text{ is a Morse function}\}.$$

Here $\varphi_n|_{\underline{X}^{n+1}}$ is the function defined by (3) and (6).

Lemma 2.6. *The set $U_{N,n}$ is a non-empty Zariski open subset of $V_{N,n}$.*

Proof. From Lemma 2.4, we have $U_{N,n} \neq \emptyset$. We define $A \subset V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C})$ as the set of $(\varphi, [Z : \mathbf{X}_1 : \dots : \mathbf{X}_{n+1}]) \in V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C})$ satisfying (5). A is a Zariski closed subset of $V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C})$. Let $i : V_{N,n} \times (\mathbb{C}^d)^{n+1} \hookrightarrow V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C})$ be the natural open immersion defined by $(\varphi, ([1 : \mathbf{x}_1], \dots, [1 : \mathbf{x}_{n+1}])) \mapsto (\varphi, [1 : \mathbf{x}_1 : \dots : \mathbf{x}_{n+1}])$. From the definition of $V_{N,n}$, we have $A \subset i(V_{N,n} \times \underline{X}^{n+1})$.

We define $B \subset V_{N,n} \times \underline{X}^{n+1}$ as the set of $(\varphi, ([1 : \mathbf{x}_1], \dots, [1 : \mathbf{x}_{n+1}])) \in V_{N,n} \times \underline{X}^{n+1}$ such that $([1 : \mathbf{x}_1], \dots, [1 : \mathbf{x}_{n+1}])$ is a degenerate critical point of $\varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}$. B is a Zariski closed subset of $V_{N,n} \times \underline{X}^{n+1}$. Since $i(B) \subset A \subset i(V_{N,n} \times \underline{X}^{n+1})$, $i(B)$ is a Zariski closed subset of A . Hence $i(B)$ is Zariski closed in $V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C})$.

Let $\pi : V_{N,n} \times \mathbb{P}^{d(n+1)}(\mathbb{C}) \rightarrow V_{N,n}$ be the natural projection. Since $\mathbb{P}^{d(n+1)}(\mathbb{C})$ is complete ([20, p. 55, Theorem 1]), $\pi(i(B))$ is Zariski closed in $V_{N,n}$. Therefore $U_{N,n} = V_{N,n} \setminus \pi(i(B))$ is Zariski open. \square

We need the following general (and standard) fact.

Lemma 2.7. *Let K be a positive integer, and let U be a non-empty Zariski open subset of \mathbb{C}^K . Then $U \cap \mathbb{R}^K$ is open dense in \mathbb{R}^K with respect to the Euclidean topology.*

Proof. $U \cap \mathbb{R}^K$ is obviously open. Note the following fact: If $f(x_1, \dots, x_K) \in \mathbb{C}[x_1, \dots, x_K]$ vanishes over a non-empty open set (of the Euclidean topology) in \mathbb{R}^K , then $f = 0$. Therefore $\mathbb{R}^K \setminus U$ cannot have an interior point in \mathbb{R}^K . Hence $U \cap \mathbb{R}^K$ is dense in \mathbb{R}^K . \square

We set $V_N^{\mathbb{R}} := V_N \cap \mathbb{R}[x_1, \dots, x_d, y_1, \dots, y_d]$ and $U_{N,n}^{\mathbb{R}} := U_{N,n} \cap \mathbb{R}[x_1, \dots, x_d, y_1, \dots, y_d]$. We naturally identify $V_N^{\mathbb{R}}$ with the Euclidean space $\mathbb{R}^{\binom{N+2d}{2d}}$. From Lemma 2.6 and Lemma 2.7, $U_{N,n}^{\mathbb{R}}$ is open dense in $V_N^{\mathbb{R}}$ with respect to the Euclidean topology. Set $U_N^{\mathbb{R}} := \bigcap_{n \geq 1} U_{N,n}^{\mathbb{R}}$. $U_N^{\mathbb{R}}$ is residual (and hence dense) in $V_N^{\mathbb{R}}$ with respect to the Euclidean topology. If $\varphi \in U_N^{\mathbb{R}}$, then for all $n \geq 1$ the functions $\varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}$ are Morse functions. Recall that we have identified M with $\underline{X}^{\mathbb{R}} = \underline{X} \cap \mathbb{R}^d$. Hence the above implies that for all $n \geq 1$ the functions $\varphi_n|_{M^{n+1}} : M^{n+1} \rightarrow \mathbb{R}$ are Morse functions.

Then we can prove Theorem 1.1:

Proof of Theorem 1.1. We define a set $\mathcal{D} \subset C^\infty(M \times M)$ by

$$(7) \quad \mathcal{D} := \bigcup_{N \geq 1} \{\varphi|_{M \times M} \mid \varphi \in U_N^{\mathbb{R}}\}.$$

We will show that \mathcal{D} is dense in $C^\infty(M \times M)$ and that it satisfies the conditions (i) and (ii) in Theorem 1.1. The condition (i) immediately follows from the above argument.

Let $f \in C^\infty(M \times M)$, and let W be an open neighborhood of f in $C^\infty(M \times M)$. There exists a real polynomial $\phi \in \mathbb{R}[x_1, \dots, x_d, y_1, \dots, y_d]$ such that $\phi|_{M \times M} \in W$ (Weierstrass's theorem). Set $N := \max(1, \deg \phi)$. We have $\phi \in V_N^{\mathbb{R}}$. Since $U_N^{\mathbb{R}}$ is dense in $V_N^{\mathbb{R}}$, there exists $\varphi \in U_N^{\mathbb{R}}$ such that $\varphi|_{M \times M} \in W$. This shows that \mathcal{D} is dense in $C^\infty(M \times M)$.

Next we want to show that \mathcal{D} satisfies the condition (ii). Let $\varphi \in U_N^{\mathbb{R}}$, and let n be a positive integer. Critical points of $\varphi_n|_{M^{n+1}} : M^{n+1} \rightarrow \mathbb{R}$ are also critical points of $\varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}$. Hence

$$(8) \quad \#\text{Cr}(\varphi_n|_{M^{n+1}}) \leq \#\{[Z : \mathbf{X}_1 : \dots : \mathbf{X}_{n+1}] \in \mathbb{P}^{d(n+1)}(\mathbb{C}) \mid \text{the condition (5)}\}.$$

From the definition of $U_N^{\mathbb{R}}$, the condition (5) implies $Z \neq 0$. Moreover all critical points of $\varphi_n|_{\underline{X}^{n+1}} : \underline{X}^{n+1} \rightarrow \mathbb{C}$ are non-degenerate. Hence the right-hand-side of (8) is finite. We can evaluate it by using Bézout's theorem [8, p. 148, Example 8.4.7] (i.e. by counting the degrees of the equations). The condition $\rho_{n,k}(Z, \mathbf{X}_1, \dots, \mathbf{X}_{n+1}) \leq r$ is equivalent to the condition that all $(r+1)$ -th sub-determinants of the matrix (4) are zero. Set $A := \max(N, \deg F_1, \dots, \deg F_R)$. By Bézout's theorem, the right-hand-side of (8) is bounded by

$$\left(A^{R+(r+1)} \binom{d}{r+1} \binom{R+1}{r+1} \right)^{n+1}.$$

(We can directly evaluate $\#\mathrm{Cr}(\varphi_n|_{M^{n+1}})$ by [18, Theorem 2] instead of Bézout's theorem, although the argument of [18] also uses Bézout's theorem.) \square

2.3. Proof of Proposition 2.1. Let M be a compact connected C^∞ manifold without boundary. We will prove Proposition 2.1 in this subsection. We need the following proposition. Professor Masahiro Shiota kindly explained this result to the authors. Probably Proposition 2.8 and its proof are well-known to some specialists of real algebraic geometry. For example, Kucharz [16, p. 128] describes the sketch of the proof of almost the same result.

Proposition 2.8. *There exist homogeneous polynomials $G_i(X_0, X_1, \dots, X_e)$ ($1 \leq i \leq S$) in $\mathbb{R}[X_0, X_1, \dots, X_e]$ satisfying the following conditions.*

- (i) *The scheme $\mathrm{Proj}(\mathbb{R}[X_0, X_1, \dots, X_e]/(G_1, \dots, G_S))$ is an integral regular scheme.*
- (ii) *The space*

$$\{[X_0 : X_1 : \dots : X_e] \in \mathbb{P}^e(\mathbb{R}) \mid G_i(X_0, X_1, \dots, X_e) = 0 \ (1 \leq i \leq S)\}$$

is diffeomorphic to M .

Proof. The idea of the proof is the same as [16, p. 128]. From Nash-Tognoli-King's theorem ([15] and [5, Chapter 14, Remark 14.1.12]), there exists a nonsingular real algebraic set $V \subset \mathbb{P}^p(\mathbb{R})$ such that V is diffeomorphic to M . (For the meaning of the term “nonsingular real algebraic set”, see [5, Section 3.3].) Since $V \cong M$ is connected, it is also Zariski connected. Hence V is irreducible in Zariski topology. (Since V is nonsingular, irreducible components of V do not intersect with each other. Hence every irreducible component of V is Zariski open and Zariski closed. See [5, Theorem 2.8.3, Proposition 3.3.10].)

Let $I \subset \mathbb{R}[X_0, X_1, \dots, X_p]$ be the homogeneous ideal generated by homogeneous polynomials $f \in \mathbb{R}[X_0, X_1, \dots, X_p]$ vanishing on V . Since V is irreducible, I is a prime ideal. Set $X := \mathrm{Proj}(\mathbb{R}[X_0, X_1, \dots, X_p]/I)$. X is an integral scheme over \mathbb{R} . From [11, p. 132, Main theorem I], there is a closed subscheme D of X satisfying the following two conditions (a) and (b):

- (a) The set of points of D is equal to the set of singular points of X .
- (b) If $m : \tilde{X} \rightarrow X$ is the monoidal transformation of X with center D , then \tilde{X} is an integral regular scheme over \mathbb{R} .

$m|_{\tilde{X} \setminus m^{-1}(D)} : \tilde{X} \setminus m^{-1}(D) \rightarrow X \setminus D$ is (algebraically) isomorphic. For generalities on monoidal transformation (or blowing-up), see Hironaka [11, pp. 123-130] and Hartshorne [10, pp. 160-169]. Let $X(\mathbb{R})$ (resp. $\tilde{X}(\mathbb{R})$) be the set of \mathbb{R} -morphisms $\mathrm{Spec} \mathbb{R} \rightarrow X$ (resp. $\mathrm{Spec} \mathbb{R} \rightarrow \tilde{X}$). The images of all \mathbb{R} -morphisms $\mathrm{Spec} \mathbb{R} \rightarrow X$ are regular points of X . Hence $X(\mathbb{R}) \cap D = \emptyset$. Therefore the natural map $\tilde{X}(\mathbb{R}) \rightarrow X(\mathbb{R})$ is a diffeomorphism. ($X(\mathbb{R})$ and $\tilde{X}(\mathbb{R})$ naturally become C^∞ manifolds.) In particular they are both diffeomorphic to $V \cong M$.

Since X is projective over \mathbb{R} , \tilde{X} is also projective over \mathbb{R} . Hence there is a homogeneous ideal $J \subset \mathbb{R}[X_0, X_1, \dots, X_e]$ such that \tilde{X} is isomorphic to $\text{Proj}(\mathbb{R}[X_0, X_1, \dots, X_e]/J)$ over \mathbb{R} . Let G_1, \dots, G_S be homogeneous polynomials generating J . Then these polynomials satisfy the above conditions (i) and (ii). \square

Set $d := (e+1)^2 - 1$. Let $\mathbb{R}[X_{00}, \dots, X_{ee}]$ be the polynomial ring of the $d+1$ variables X_{ij} ($0 \leq i, j \leq e$). Consider a \mathbb{R} -homomorphism from $\mathbb{R}[X_{00}, \dots, X_{ee}]$ to $\mathbb{R}[X_0, \dots, X_e]$ defined by

$$X_{00} \mapsto \sum_{i=0}^e X_i^2, \quad X_{ij} \mapsto X_i X_j \quad ((i, j) \neq (0, 0)).$$

Let $f : \text{Proj}(\mathbb{R}[X_0, \dots, X_e]) \rightarrow \text{Proj}(\mathbb{R}[X_{00}, \dots, X_{ee}])$ be the \mathbb{R} -morphism defined by the above homomorphism. The map f is a closed immersion (cf. Segre embedding). Moreover f satisfies

$$f(\mathbb{P}^e(\mathbb{R})) \subset \mathbb{R}^d := \{[X_{00} : \dots : X_{ee}] \in \mathbb{P}^d(\mathbb{R}) \mid X_{00} \neq 0\}.$$

(This fact is used in [5, p. 72] to show that real projective spaces are affine varieties.) From this argument and Proposition 2.8 we get the following:

Corollary 2.9. *There exist homogeneous polynomials $F_i(X_0, X_1, \dots, X_d)$ ($1 \leq i \leq R$) in $\mathbb{R}[X_0, X_1, \dots, X_d]$ satisfying the following conditions (i) and (ii):*

- (i) $\text{Proj}(\mathbb{R}[X_0, X_1, \dots, X_d]/(F_1, \dots, F_R))$ is an integral regular scheme.
- (ii) The space

$$\{[X_0 : X_1 : \dots : X_d] \in \mathbb{P}^d(\mathbb{R}) \mid F_i(X_0, X_1, \dots, X_d) = 0 \ (1 \leq i \leq R)\}$$

is diffeomorphic to M . Moreover, if a point $[X_0 : X_1 : \dots : X_d] \in \mathbb{P}^d(\mathbb{R})$ satisfies $F_i(X_0, \dots, X_d) = 0$ ($1 \leq i \leq R$), then $X_0 \neq 0$.

Proof of Proposition 2.1. Let $F_i(X_0, X_1, \dots, X_d)$ ($1 \leq i \leq R$) be the polynomials introduced in Corollary 2.9. The condition (i) of Corollary 2.9 implies that the scheme

$$\text{Proj}(\mathbb{C}[X_0, \dots, X_d]/(F_1, \dots, F_R)) = \text{Proj}(\mathbb{R}[X_0, \dots, X_d]/(F_1, \dots, F_R)) \times_{\mathbb{R}} \mathbb{C}$$

is an equidimensional regular scheme.

Let X be the complex submanifold of $\mathbb{P}^d(\mathbb{C})$ which F_i ($1 \leq i \leq R$) define. From Bertini's theorem (Hartshorne [10, p. 179, Theorem 8.18]), there exists a non-empty Zariski open set $U \subset \mathbb{P}^d(\mathbb{C})$ such that for any $[a_0 : a_1 : \dots : a_d] \in U$ the hyperplane $a_0 X_0 + a_1 X_1 + \dots + a_d X_d = 0$ transversally intersects with X in $\mathbb{P}^d(\mathbb{C})$. $U \cap \mathbb{P}^d(\mathbb{R})$ is open dense in $\mathbb{P}^d(\mathbb{R})$ with respect to the Euclidean topology. (See Lemma 2.7.)

We have $X \cap \mathbb{R}^d = X \cap \mathbb{P}^d(\mathbb{R})$. Hence $X \cap \mathbb{R}^d$ is compact. Therefore there exists $[a_0 : a_1 : \dots : a_d] \in U \cap \mathbb{P}^d(\mathbb{R})$ such that the hyperplane $a_0 X_0 + a_1 X_1 + \dots + a_d X_d = 0$ does not intersect with $X \cap \mathbb{R}^d$. Then, by using a real projective transformation which transforms $a_0 X_0 + a_1 X_1 + \dots + a_d X_d = 0$ to $X_0 = 0$, we can adjust the polynomials F_i so that they satisfy the conditions (i), (ii), (iii) in Proposition 2.1. \square

3. PROOF OF THEOREM 1.3

In this section, for a C^∞ manifold M (not necessarily compact), the space $C^\infty(M)$ of real valued C^∞ functions on M is endowed with the C^∞ compact-open topology.

3.1. Interpretation to a dynamical problem. For a C^∞ function H on \mathbb{R}^2 , we denote the partial derivative of H with respect to the first and the second coordinates by $\partial_1 H$ and $\partial_2 H$ respectively. By p_1 and p_2 , we denote the projection from \mathbb{R}^2 to the first and the second coordinate, respectively. We say that a map f from \mathbb{R}^2 to a set S is $(2\pi\mathbb{Z})^2$ -periodic if $f(p+m) = f(p)$ for any $p \in \mathbb{R}^2$ and $m \in (2\pi\mathbb{Z})^2$. Let \mathcal{H} be the space of C^∞ functions H on \mathbb{R}^2 such that $\partial_1 \partial_2 H > 0$, and both $\partial_1 H - p_2$ and $\partial_2 H - p_1$ are $(2\pi\mathbb{Z})^2$ -periodic. We denote the identity map of \mathbb{R}^2 by Id and define a diffeomorphism Θ of \mathbb{R}^2 by $\Theta(x, y) = (y, -x)$. Let \mathcal{D} be the set of C^∞ area-preserving diffeomorphisms F of \mathbb{R}^2 such that $F - \Theta$ is $(2\pi\mathbb{Z})^2$ -periodic and *the twist condition*

$$(9) \quad \partial_2(p_1 \circ F) > 0$$

holds. We endow the C^∞ compact-open topology to \mathcal{D} .

The following is a classical result known as the correspondence between twist maps and their generating functions.

Proposition 3.1. *There exists a continuous map $\Phi : \mathcal{H} \rightarrow \mathcal{D}$ which satisfies the following properties.*

- (i) $(x', y') = \Phi(H)(x, y)$ if and only if $(y, y') = (\partial_1 H(x, x'), -\partial_2 H(x, x'))$ for any $H \in \mathcal{H}$ and $(x, y, x', y') \in \mathbb{R}^4$.
- (ii) The map $(\Phi, \text{ev}_0) : \mathcal{H} \rightarrow \mathcal{D} \times \mathbb{R}$ is a homeomorphism, where the map $\text{ev}_0 : \mathcal{H} \rightarrow \mathbb{R}$ is given by $\text{ev}_0(H) = H(0, 0)$.

Proof. For $H \in \mathcal{H}$, we define two maps $\phi_H, \psi_H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{aligned} \phi_H(x, x') &= (x, \partial_1 H(x, x')) \\ \psi_H(x, x') &= (x', -\partial_2 H(x, x')). \end{aligned}$$

Put $g_x(x') = \partial_1 H(x, x')$. Since $dg_x/dx' = \partial_2 \partial_1 H > 0$ and $g_x(x' + m) = g_x(x') + m$ for any $x' \in \mathbb{R}$ and $m \in 2\pi\mathbb{Z}$, the map $g_x(x') = \partial_1 H(x, x')$ is a diffeomorphism of \mathbb{R} for any $x \in \mathbb{R}$. Hence, ϕ_H is a diffeomorphism of \mathbb{R}^2 . Similarly, so is ψ_H . We define a diffeomorphism $\Phi(H)$ of \mathbb{R}^2 by $\Phi(H) = \psi_H \circ \phi_H^{-1}$. Since the maps $\phi_H - \text{Id}$ and $\psi_H - \Theta$ are $(2\pi\mathbb{Z})^2$ -periodic, $\Phi(H) - \Theta$ is also $(2\pi\mathbb{Z})^2$ -periodic. By direct computation, we can see that $\Phi(H)$ is area-preserving and satisfies the twist condition (9). Therefore, $\Phi(H)$ is an element of \mathcal{D} .

For $F \in \mathcal{D}$, we define two maps $\hat{\phi}_F, \hat{\psi}_F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{aligned} \hat{\phi}_F(x, y) &= (x, p_1 \circ F(x, y)) \\ \hat{\psi}_F(x', y') &= (p_1 \circ F^{-1}(x', y'), x'). \end{aligned}$$

By the twist condition (9), $\hat{\phi}_F$ is a diffeomorphism. For $(x', y') = F(x, y)$, we have $\hat{\phi}_F(x, y) = \hat{\psi}_F \circ F(x, y) = (x, x')$. Hence, $\hat{\psi}_F$ is a diffeomorphism and $F = \hat{\psi}_F^{-1} \circ \hat{\phi}_F$. Take $g_1, g_2 \in C^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}\hat{\phi}_F^{-1}(x, x') &= (x, g_1(x, x')) \\ \hat{\psi}_F^{-1}(x, x') &= (x', -g_2(x, x')).\end{aligned}$$

By a direct calculation, we obtain that

$$\begin{aligned}\partial_1 g_2 - \partial_2 g_1 &= \log \det DF_{\hat{\phi}_F^{-1}(x, x')} = 0, \\ (\partial_2 g_1)^{-1} &= \partial_2(p_1 \circ F)(\hat{\phi}_F^{-1}(x, x')) > 0.\end{aligned}$$

The former implies that the one-form $g_1 dx + g_2 dx'$ is closed. By Poincaré's lemma, there exists a unique C^∞ -function H_F such that $H_F(0, 0) = 0$ and $dH_F = g_1 dx + g_2 dx'$. Since $F - \Theta$ is $(2\pi\mathbb{Z})^2$ -periodic, the maps $\hat{\phi}_F - \text{Id}$ and $\hat{\psi}_F - \Theta^{-1}$ are $(2\pi\mathbb{Z})^2$ -periodic. This implies that $\hat{\phi}_F^{-1} - \text{Id}$ and $\hat{\psi}_F^{-1} - \Theta$ are $(2\pi\mathbb{Z})^2$ -periodic, and hence, so are $g_1 - p_2$ and $g_2 - p_1$. Therefore, H_F is an function in \mathcal{H} . We define a map $\Psi : \mathcal{D} \times \mathbb{R} \rightarrow \mathcal{H}$ by $\Psi(F, c) = H_F + c$. Since

$$F = \hat{\psi}_F^{-1} \circ \hat{\phi}_F = \psi_{H_F} \circ \phi_{H_F}^{-1} = \Phi(H_F),$$

Ψ is the inverse of (Φ, ev_0) . The continuity of Φ and Ψ follows from the constrictions. \square

Corollary 3.2. *For $H \in \mathcal{H}$ and $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, the following two conditions are equivalent.*

- (i) (x_0, \dots, x_n) is a critical point of $H_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ where $H_n(x_0, \dots, x_n) := \sum_{i=0}^{n-1} H(x_i, x_{i+1})$.
- (ii) There exists $(y_0, \dots, y_n) \in \mathbb{R}^{n+1}$ such that $y_0 = y_n = 0$ and $(x_{j+1}, y_{j+1}) = \Phi(H)(x_j, y_j)$ for any $j = 0, \dots, n-1$.

Proof. For $H \in C^\infty(\mathbb{R}^2)$, a point $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ is a critical point of H_n if and only if $\partial_1 H(x_0, x_1) = \partial_2 H(x_{n-1}, x_n) = 0$ and $\partial_1 H(x_j, x_{j+1}) = -\partial_2 H(x_{j-1}, x_j)$ for any $j = 1, \dots, n-1$. Hence, the corollary follows from Proposition 3.1. \square

Hence, the counting of critical point of H_n is reduced to the counting of points in $\Phi(H)^{-n}(\mathbb{R} \times \{0\}) \cap (\mathbb{R} \times \{0\})$.

3.2. Abundance of recurrence of intervals. Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the two-dimensional torus. Set $M = \mathbb{R}^2$ or \mathbb{T}^2 . By $\text{Diff}_\omega(M)$, we denote the set of area-preserving diffeomorphisms of M endowed with the C^∞ compact-open topology. By $\text{Int } I$, we denote the interior of an interval I . For embedded intervals I and J in M , let $I \pitchfork J$ be the set of transverse intersections of $\text{Int } I$ and $\text{Int } J$.

Let us recall some definitions and known facts on dynamical systems. The authors recommend [14] or [22] for reference. For $f \in \text{Diff}_\omega(M)$, a fixed point p of f^N is called *hyperbolic* if no eigenvalues of Df_p^N is of absolute value one. Remark that one of the eigenvalues is of absolute value greater than one and the other is less than one since f is

area-preserving. A *continuation* of a hyperbolic fixed point p of f^N is a continuous map \hat{p} from a neighborhood $\mathcal{U} \subset \text{Diff}_\omega(M)$ of f to M such that $\hat{p}(f) = p$ and $\hat{p}(g)$ is a hyperbolic fixed point of g^N for any $g \in \mathcal{U}$. It is known that any hyperbolic fixed point admits a continuation.

Let d be the standard distance on $M = \mathbb{R}^2$ or \mathbb{T}^2 . For a hyperbolic fixed point p of f^N , the *stable manifold* $W^s(p; f)$ and the *unstable manifold* $W^u(p; f)$ are defined by

$$\begin{aligned} W^s(p; f) &= \{q \in M \mid d(f^n(p), f^n(q)) \rightarrow 0 \ (n \rightarrow +\infty)\}, \\ W^u(p; f) &= \{q \in M \mid d(f^n(p), f^n(q)) \rightarrow 0 \ (n \rightarrow -\infty)\}. \end{aligned}$$

By the stable manifold theorem, both $W^s(p; f)$ and $W^u(p; f)$ are C^∞ injectively immersed curves. For $L > 0$, let $W_L^s(p; f)$ and $W_L^u(p; f)$ be the compact subintervals of $W^s(p; f)$ and $W^u(p; f)$ centered at p whose length is $2L$. The set $W_L^s(p; f)$ satisfies $f^N(W_L^s(p; f)) \subset W_L^s(p; f)$ and $W^s(p; f) = \bigcup_{k \geq 0} f^{-kN} W_L^s(p; f)$. The set $W_L^u(p; f)$ also have similar properties. For a continuation $\hat{p} : \mathcal{U} \rightarrow M$ of p , it is known that $W_L^s(\hat{p}(f); f)$ and $W_L^u(\hat{p}(f); f)$ depends continuously on f as C^∞ embedded intervals.

We say that a hyperbolic fixed point p of f^N *exhibits homoclinic tangency* at a point $q \in W^s(p; f) \cap W^u(p; f)$ if $W^s(p; f)$ and $W^u(p; f)$ are tangent at q . We also say that the homoclinic tangency at q is ∞ -flat if the ∞ -jets of $W^s(p; f)$ and $W^u(p; f)$ at q coincide.

The aim of this section is to show the following.

Proposition 3.3. *Let J be an interval in \mathbb{T}^2 , f_0 a diffeomorphism in $\text{Diff}_\omega(\mathbb{T}^2)$, and p_0 a hyperbolic fixed point of f_0^N for some $N \geq 1$. Suppose that p_0 exhibits homoclinic tangency and $W^\sigma(p_0; f_0) \cap J \neq \emptyset$ for each $\sigma = s, u$. Then, there exists an open subset \mathcal{U}_* of $\text{Diff}_\omega(\mathbb{T}^2)$ such that $f_0 \in \overline{\mathcal{U}_*}$ and the set*

$$\mathcal{U}_n = \bigcup_{m \geq n} \{f \in \mathcal{U}_* \mid \# [f^m(J) \cap J] \geq a_m\}$$

is an open dense subset of \mathcal{U}_ for any given sequence $(a_m)_{m \geq 1}$ of positive integers and any $n \geq 1$.*

The first ingredient of the proof is *The Inclination Lemma* (or *The λ -lemma*). See e.g. [22, Theorem V.11.1] for the proof.

Theorem 3.4. *Let p be a hyperbolic fixed point of $f \in \text{Diff}_\omega(\mathbb{T}^2)$, I and J embedded compact intervals in \mathbb{T}^2 , and K a closed subset of \mathbb{T}^2 such that $p \in I \subset W^u(p; f)$, $J \cap W^s(p; f)$ contains a point z , and $K \cap [I \cup \{f^n(z); n \geq 0\}] = \emptyset$. Then, there exists a sequence $(J_k)_{k \geq 1}$ of subintervals of J such that*

- (i) $f^n(J_k) \cap K = \emptyset$ for any $n = 0, \dots, k$.
- (ii) $f^k(J_k)$ converges to I as a C^∞ embedded interval as $k \rightarrow \infty$.

Take a neighborhood $\mathcal{U}_0 \subset \text{Diff}_\omega(\mathbb{T}^2)$ of f_0 and a continuation $\hat{p} : \mathcal{U}_0 \rightarrow \mathbb{T}^2$ of p_0 . Applying The Inclination Lemma, we give a criterion to approximation by a diffeomorphism g such that $g^m(J) \cap J$ contains infinitely many points.

Lemma 3.5. *Let J^- and J^+ be compact intervals in \mathbb{T}^2 , f a diffeomorphism in $\text{Diff}_\omega(\mathbb{T}^2)$, and p a hyperbolic fixed point of f^N with some $N \geq 1$. Suppose that there exist $L > 0$ and $q \in [W^s(p; f) \setminus W_L^s(p; f)] \cap [W^u(p; f) \setminus W_L^u(p; f)]$ such that $J^- \cap W_L^s(p; f) \neq \emptyset$, $J^+ \cap W_L^u(p; f) \neq \emptyset$, and p exhibits ∞ -flat homoclinic tangency at q . Then, for any give neighborhood \mathcal{U} of f in $\text{Diff}_\omega(\mathbb{T}^2)$ and any $n_0 \geq 1$, there exists a diffeomorphism $g \in \mathcal{U}$ and $n_* \geq n_0$ such that*

$$\# [g^{n_*}(J^-) \cap J^+] = \infty.$$

Proof. We may assume that N is the minimal period of p , i.e., the minimal positive integer satisfying $f^N(p) = p$. Let m_- and m_+ be the minimal positive integers such that $f^{-m_-}(q) \in W_L^u(p; f)$ and $f^{m_+}(q) \in W_L^s(p; f)$. Since $q \in W^s(p; f) \cap W^u(p; f)$ and $W^\sigma(p; f) \cap W^\sigma(f^j(p); f) = \emptyset$ for $\sigma = s, u$ if $j \not\equiv 0 \pmod{N}$, we have $m_\pm \geq N$. Take $L' > L$ such that $f^j(q) \notin W_{L'}^u(p; f) \cup W_{L'}^s(p; f)$ for any $j = -m_- + 1, \dots, m_+ - 1$. We also take an open neighborhood U of q such that $\overline{f^i(U)} \cap [W_{L'}^u(p; f) \cup W_{L'}^s(p; f)] = \emptyset$ and $f^i(U) \cap f^j(U) = \emptyset$ for any $i, j = -m_- + 1, \dots, m_+ - 1$ with $i \neq j$.

By The Inclination Lemma, there exists a sequence $(J_k^-)_{k \geq 1}$ of subintervals of J^- such that $f^{iN}(J_k^-) \cap \bigcup_{j=-m_-+1}^{m_+-1} f^j(U) = \emptyset$ for any $i = 0, \dots, k$ and $f^{kN}(J_k^-)$ converges to $W_{L'}^u(p; f)$ as $k \rightarrow \infty$. Since $m_\pm \geq N$, the former implies that $f^j(J_k^-) \cap U = \emptyset$ for any $j = 0, \dots, kN + m_- - 1$. Take an interval I^u in $f^{m_-}(W_{L'}^u(p; f)) \cap U$ such that $\text{Int } I^u$ contains q . Then, there exists a sequence $(I_k^-)_{k \geq 1}$ of subintervals of J^- such that $I_k^- \subset J_k^-$ for any $k \geq 1$ and $f^{kN+m_-}(I_k^-)$ converges to I^u as $k \rightarrow \infty$. It satisfies that $f^j(I_k^-) \cap U = \emptyset$ for any $j = 0, \dots, kN + m_- - 1$ and $f^{kN+m_-}(I_k^-)$ converges to I^u as $k \rightarrow \infty$.

Take an interval I^s in $f^{-m_+}(W_{L'}^s(p; f)) \cap U$ such that $\text{Int } I^s$ contains q . By the same argument as above, we can take a sequence $(I_k^+)_{k \geq 1}$ of subintervals J^+ such that $f^{-j}(I_k^+) \cap U = \emptyset$ for any $j = 0, \dots, kN + m_+ - 1$ and $f^{-(kN+m_+)}(I_k^+)$ converges to I^s as $k \rightarrow \infty$.

Fix an neighborhood \mathcal{U} of f in $\text{Diff}_\omega(\mathbb{T}^2)$ and an integer $n_0 \geq 1$. Let \mathcal{V} be the set of diffeomorphisms $\phi \in \text{Diff}_\omega(\mathbb{T}^2)$ such that $f \circ \phi^{-1} \in \mathcal{U}$ and $\text{supp}(\phi) \in U$. Since I^s and I^u are compact intervals in U and they have the same ∞ -jets at q , there exists $k_* \geq n_0$ and $\phi_1 \in \mathcal{V}$ such that the set $\phi_1(f^{-(k_*N+m_+)}(I_{k_*}^+)) \cap f^{k_*N+m_-}(I_{k_*}^-)$ contains an interval. Put $n_+ = k_*N + m_+$ and $n_- = k_*N + m_-$. Take a small perturbation $\phi_2 \in \mathcal{V}$ of ϕ_1 such that $\# [\phi_2(f^{-n_+}(I_{k_*}^+)) \cap f^{n_-}(I_{k_*}^-)] = \infty$. Put $g = f \circ \phi_2^{-1} \in \mathcal{U}$. It is easy to check that $g^{n_-}(I_{k_*}^-) = f^{n_-}(I_{k_*}^-)$ and $g^{-n_+}(I_{k_*}^+) = \phi_2 \circ f^{-n_+}(I_{k_*}^+)$. Therefore, we have $\# [g^{n_+}g^{n_-}(I_{k_*}^-) \cap I_{k_*}^+] = \infty$. Since $k_* \geq n_0$, we also have $n_+ + n_- \geq n_0$. \square

The other ingredients are the following results on homoclinic tangency.

Theorem 3.6 (Duarte [6]). *Let f_0 be a diffeomorphism in $\text{Diff}_\omega(\mathbb{T}^2)$, p_0 a hyperbolic fixed point of f^N , and $\hat{p} : \mathcal{U}_0 \rightarrow \mathbb{T}^2$ a continuation of p_0 on an open neighborhood \mathcal{U}_0 of f_0 . If p_0*

exhibits homoclinic tangency, then there exists an open set $\mathcal{U} \subset \mathcal{U}_0$ and a dense subset \mathcal{T} of \mathcal{U} such that $f_0 \in \overline{\mathcal{U}}$ and $\hat{p}(f)$ exhibits homoclinic tangency for any $f \in \mathcal{T}$.

Theorem 3.7 (Gonchenko-Turaev-Shilnikov [9]). *Let f_0 be a diffeomorphism in $\text{Diff}_\omega(\mathbb{T}^2)$ and p_0 a hyperbolic fixed point of f_0^N . If p_0 exhibits homoclinic tangency, then any neighborhood of f_0 contains a diffeomorphism g such that p_0 is a hyperbolic fixed point of g^N and it exhibits ∞ -flat homoclinic tangency.*

Now, we prove Proposition 3.3. Let J , f_0 , and p_0 be the ones in the assumption of the proposition. Take $L > 0$, an open neighborhood \mathcal{U}_0 of f_0 and a continuation $\hat{p} : \mathcal{U}_0 \rightarrow \mathbb{T}^2$ of p_0 such that $W_L^s(\hat{p}(f); f) \cap J \neq \emptyset$ and $W_L^u(\hat{p}(f); f) \cap J \neq \emptyset$ for any $f \in \mathcal{U}_0$. By the Kupka-Smale Theorem, $W^s(\hat{p}(f); f)$ and $W^u(\hat{p}(f); f)$ intersect transversely for generic $f \in \mathcal{U}_0$. Hence, there exists an open and dense subset \mathcal{U}_1 of \mathcal{U}_0 such that $f \in \overline{\mathcal{U}_1}$ and all intersections of $W_L^s(\hat{p}(f); f)$ and $W_L^u(\hat{p}(f); f)$ are transverse for any $f \in \mathcal{U}_1$. For $n \geq 1$, put

$$\mathcal{T}_n = \{f \in \text{Diff}_\omega(\mathbb{T}^2) \mid \# [f^n(J) \cap J] = \infty\}.$$

By Theorems 3.6 and 3.7 and Lemma 3.5, we can take an open subset \mathcal{U}_* of \mathcal{U}_1 such that $f_0 \in \overline{\mathcal{U}_*}$ and the set $(\bigcup_{m \geq n} \mathcal{T}_m) \cap \mathcal{U}_*$ is dense in \mathcal{U}_* for any $n \geq 1$. This implies that the set $\mathcal{U}_n = \bigcup_{m \geq n} \{f \in \mathcal{U}_* \mid \# [f^m(J) \cap J] \geq a_m\}$ is an open and dense subset of \mathcal{U}_* for any sequence $(a_m)_{m \geq 1}$ and any $n \geq 1$.

3.3. Proof of Theorem 1.3. In this subsection, we prove Theorem 1.3. Recall that \mathcal{D} is the set of diffeomorphisms $F \in \text{Diff}_\omega(\mathbb{R}^2)$ such that $F - \Theta$ is $(2\pi\mathbb{Z})^2$ -periodic and the twist condition (9) holds, where $\Theta(x, y) = (y, -x)$. Let $\pi_T : \mathbb{R}^2 \rightarrow \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the natural projection. It induces a map $\pi_{T*} : \mathcal{D} \rightarrow \text{Diff}_\omega(\mathbb{T}^2)$. For a diffeomorphism $F \in \mathcal{D}$, an open subset U of \mathbb{R}^2 , and $n \geq 1$, we put

$$\Lambda^n(F, U) = [(\mathbb{R} \times \{0\}) \cap F^{-n}(\mathbb{R} \times \{0\})] \cap \bigcap_{m=0}^n F^{-m}(U).$$

First, we “lift” Proposition 3.3 to the set \mathcal{D} .

Proposition 3.8. *Suppose that a diffeomorphism F_0 in \mathcal{D} , a hyperbolic fixed point p_0 of F_0^N ($N \geq 1$), open subsets U_0 and U_1 of \mathbb{R}^2 , and open subset \mathcal{U}_0 of \mathcal{D} satisfy the following four conditions:*

- (i) $p_0 \in U_0 \subset U_1$, $\overline{U_1} \subset (-\pi, \pi)^2$, and $F_0 \in \mathcal{U}_0$.
- (ii) p_0 exhibits homoclinic tangency.
- (iii) $W^\sigma(p_0; F_0) \cap [(\mathbb{R} \times \{0\}) \cap U_0] \neq \emptyset$ for $\sigma = s, u$.
- (iv) $F^n(U_0) \subset U_1$ for any $F \in \mathcal{U}_0$ and $n \in \mathbb{Z}$.

Then, there exists an open subset \mathcal{U} of \mathcal{U}_0 such that $F_0 \in \overline{\mathcal{U}}$ and the set

$$\bigcup_{m \geq n} \{F \in \mathcal{U} \mid \# \Lambda^m(F, U_1) \geq a_m\}$$

is open and dense in \mathcal{U} for any given sequence $(a_m)_{m \geq 1}$ of positive integers and $n \geq 1$.

Proof. Put $f_0 = \pi_{T*}(F_0)$, $p_T = \pi_T(p_0)$, $L_0 = (\mathbb{R} \times \{0\}) \cap U_0$, and $L_T = \pi_T(L_0)$. The hyperbolic fixed point p_T of f_0^N exhibits homoclinic tangency. By assumption, $W^\sigma(p_T; f_0) \cap L_T \neq \emptyset$ for $\sigma = s, u$. Applying Proposition 3.3, we obtain an open subset \mathcal{U}_T of $\text{Diff}_\omega(\mathbb{T}^2)$ such that $f_0 \in \overline{\mathcal{U}_T}$ and the set

$$\mathcal{U}_n = \bigcup_{m \geq n} \{f \in \mathcal{U}_T \mid \# [L_T \cap f^{-m}(L_T)] \geq a_m\}$$

is open and dense in \mathcal{U}_T for any sequence $(a_m)_{m \geq 1}$ of positive integers and $n \geq 1$.

The set $\pi_{T*}(\mathcal{D})$ is an open subset of $\text{Diff}_\omega(\mathbb{T}^2)$, and π_{T*} is a covering map onto $\pi_{T*}(\mathcal{D})$. Hence, there exists an open subset \mathcal{U} of \mathcal{U}_0 such that $F_0 \in \overline{\mathcal{U}}$, the restriction of π_{T*} to \mathcal{U} is a homeomorphism onto an open subset of \mathcal{U}_T . Since $L_0 = \pi_T^{-1}(L_T) \cap U_0$ and $F^n(U_0) \subset U_1$ for any $F \in \mathcal{U}$ and $n \in \mathbb{Z}$, we have $F^n(L_0) \subset U_1$. This implies that $L_0 \cap F^{-n}(L_0) \subset \bigcap_{m=0}^n F^{-m}(U_1)$, and hence,

$$\#\Lambda^n(F, U_1) \geq \# [L_0 \cap F^{-n}(L_0)] = \# [L_T \cap (\pi_{T*}(F))^{-n}(L_T)].$$

Since π_{T*} maps \mathcal{U} to an open subset of \mathcal{U}_T homeomorphically, the set

$$\bigcup_{m \geq n} \{F \in \mathcal{U} \mid \#\Lambda^m(F, U_1) \geq a_m\}$$

is open and dense in \mathcal{U} for any given sequence $(a_m)_{m \geq 1}$ of positive integers and any $n \geq 1$. \square

Next, we see that the existence of a diffeomorphism F_0 satisfying Proposition 3.8 implies Theorem 1.3. Let U_0 and \mathcal{U} be open subsets of $(-\pi, \pi)^2$ and \mathcal{D} in Proposition 3.8. Put $\mathcal{U}_H = \Phi^{-1}(\mathcal{U})$, where $(\Phi, \text{ev}_0) : \mathcal{H} \rightarrow \mathcal{D} \times \mathbb{R}$ is the homeomorphism given in Proposition 3.1. Take a compact interval $I \subset (-\pi, \pi)$ such that $U_0 \subset I \times \mathbb{R}$. By Corollary 3.2,

$$\# [\text{Cr}(H_n) \cap I^{n+1}] \geq \#\Lambda^n(\Phi(H), U_0)$$

for any $H \in \mathcal{U}_H$. Fix a sequence $(a_m)_{m \geq 1}$ of positive integers. Then, the set

$$\mathcal{U}_n = \bigcup_{m \geq n} \{H \in \mathcal{U}_H \mid \# [\text{Cr}(H_m) \cap I^{m+1}] \geq a_m\}$$

contains an open dense subset of \mathcal{U}_H for any $n \geq 1$.

Let $C^\infty(I^2)$ be the set of functions on I^2 which extends to a C^∞ function on an open neighborhood of I^2 . Recall that we identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Let $r_S : C^\infty(S^1 \times S^1) \rightarrow C^\infty(I^2)$ and $r_H : \mathcal{H} \rightarrow C^\infty(I^2)$ be the maps induced by the restriction of functions. They are continuous and open. Hence, the set $\mathcal{U}_S = r_S^{-1}(r_H(\mathcal{U}_H))$ is open in $C^\infty(S^1 \times S^1)$. Moreover, if \mathcal{U}' is an open dense subset of \mathcal{U}_H , then $r_S^{-1}(r_H(\mathcal{U}'))$ is also open and dense in \mathcal{U}_S . Since

$$\# [\text{Cr}(f_n) \cap I^{n+1}] = \# [\text{Cr}(H_n) \cap I^{n+1}]$$

for $f \in \mathcal{U}_S$ and $H \in \mathcal{U}_H$ with $r_S(f) = r_H(H)$, the set

$$\mathcal{U}'_n = \bigcup_{m \geq n} \{f \in \mathcal{U}_S \mid \#[\text{Cr}(f_m) \cap I^{m+1}] \geq a_m\} \supset r_S^{-1}(r_H(\mathcal{U}_n))$$

contains an open dense subset of \mathcal{U}_S for any given sequence $(a_m)_{m \geq 1}$ of positive integers and any $n \geq 1$. Therefore, the set

$$\left\{f \in \mathcal{U}_S \mid \limsup_{n \rightarrow \infty} \frac{\#\text{Cr}(f_n)}{a_n} \geq 1\right\} \supset \bigcap_{n \geq 1} \mathcal{U}'_n$$

is residual in \mathcal{U}_S . This is just the statement of Theorem 1.3.

Finally, we construct a diffeomorphism in \mathcal{D} which satisfies the assumption of Proposition 3.8. Put $O = (0, 0)$ and $p_0 = (1, 0)$. For $r > 0$, we define a disk $D'(r)$ and an annulus $A(r)$ in \mathbb{R}^2 by

$$\begin{aligned} D'(r) &= \{q \in \mathbb{R}^2 \mid \|q - p_0\| \leq r\} \\ A(r) &= \{q \in \mathbb{R}^2 \mid 2 - r \leq \|q\| \leq 2 + r\}. \end{aligned}$$

Here $\|\cdot\|$ is the Euclidean norm. Fix $\delta = 1/6$ and let G be a $(2\pi\mathbb{Z})^2$ -periodic C^∞ function on \mathbb{R}^2 such that

$$G(x, y) = \begin{cases} (x^2 + y^2)^2 & \text{for } (x, y) \in A(\delta) \\ y^2 + (x - 1)^2(x - (1 + \delta)) & \text{for } (x, y) \in D'(2\delta), \end{cases}$$

and

$$\text{supp}(G) \cap (-\pi, \pi)^2 \subset A(2\delta) \cup D'(3\delta).$$

Let Ψ_t be the flow generated by a vector field $X_G = (\partial_2 G)\partial_1 - (\partial_1 G)\partial_2$. It is an area-preserving flow satisfying the following properties for any $t > 0$:

- (i) $\text{supp}(\Psi_t) \cap (-\pi, \pi)^2 \subset A(2\delta) \cup D'(3\delta)$.
- (ii) $\Psi_t(r \cos \theta, r \sin \theta) = (r \cos(\theta - 4r^2 t), r \sin(\theta - 4r^2 t))$ for any $(r \cos \theta, r \sin \theta) \in A(\delta)$.
- (iii) p_0 is a hyperbolic fixed point of Ψ_t .
- (iv) $\{(x, y) \in D'(2\delta) \mid x \geq 1, y^2 + (x - 1)^2(x - (1 + \delta)) = 0\}$ is contained in $W^s(p_0; \Psi_t) \cap W^u(p_0; \Psi_t)$.

Remark that the last item implies that p_0 exhibits homoclinic tangency and $W^\sigma(p_0; \Psi_t) \cap (\mathbb{R} \times \{0\}) \neq \emptyset$ for $\sigma = s, u$.

Since \mathcal{D} is an open subset of the set of diffeomorphisms $F \in \text{Diff}_\omega(\mathbb{R}^2)$ such that $F - \Theta$ is $(2\pi\mathbb{Z})^2$ -periodic, there exists small $T > 0$ such that $\Theta \circ \Psi_T \in \mathcal{D}$. Put $F_0 = \Theta \circ \Psi_T$. Since $F_0^4 = \Psi_T$ on $D'(2\delta)$, p_0 is a hyperbolic fixed point of F_0^4 and it exhibits homoclinic tangency. For $(r \cos \theta, r \sin \theta) \in A(\delta)$, we have

$$F_0^4(r \cos \theta, r \sin \theta) = \Psi_T^4(r \cos \theta, r \sin \theta) = (r \cos(\theta - 16r^2 T), r \sin(\theta - 16r^2 T)).$$

The Kolmogorov-Arnold-Moser Theorem on persistence of invariant circle (see *e.g.*, [19]) implies that there exist a neighborhood \mathcal{U}_0 of F_0 such that any $F \in \mathcal{U}_0$ admits an F -invariant circle $C(F) \subset A(\delta)$ which winds the annulus once. This implies that open disks $U_0 = \{q \in \mathbb{R}^2 \mid \|q\| < 2 - \delta\}$ and $U_1 = \{q \in \mathbb{R}^2 \mid \|q\| < 2 + \delta\}$ satisfy that $F^n(U_0) \subset U_1$ for any $F \in \mathcal{U}_0$ and $n \in \mathbb{Z}$. Therefore, F_0 and p_0 satisfy the assumption of Proposition 3.8. As we mentioned in the above, it completes the proof of Theorem 1.3.

4. PROOF OF THEOREM 1.4

We denote the natural coordinate of $(S^1)^{n+1} = (\mathbb{R}/2\pi\mathbb{Z})^{n+1}$ by (x_0, x_1, \dots, x_n) . Let $F : (S^1)^{n+1} \rightarrow \mathbb{R}$ be a smooth function. For $x \in (S^1)^{n+1}$, we set

$$\text{Hess}_x F := \left(\frac{\partial^2 F(x)}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq n}.$$

This is a $(n+1, n+1)$ -matrix. We set

$$M(F)(x) := \max_{0 \leq i \leq n} \left| \frac{\partial F(x)}{\partial x_i} \right| + \inf_{u \in \mathbb{R}^{n+1}, \|u\|=1} \|\text{Hess}_x F(u)\|.$$

F is a Morse function if and only if $M(F)(x) > 0$ for all $x \in (S^1)^{n+1}$. (Here $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^{n+1} .)

Let $f : S^1 \times S^1 \rightarrow \mathbb{R}$ be a smooth function. It is easy to check that for any $n \geq 1$

$$(10) \quad \sup_{0 \leq i \leq n} \left| \frac{\partial f_n(x)}{\partial x_i} \right| \leq 2 \|\nabla f\|_\infty, \quad \|\text{Hess}_x(f_n)u\| \leq 5 \|\nabla^2 f\|_\infty \|u\| \quad (u \in \mathbb{R}^{n+1}),$$

where $\|\nabla f\|_\infty$ is the supremum of $|\partial f(x)/\partial x_i|$ over $x \in S^1 \times S^1$ and $0 \leq i \leq 1$, and $\|\nabla^2 f\|_\infty$ is the supremum of $|\partial^2 f(x)/\partial x_i \partial x_j|$ over $x \in S^1 \times S^1$ and $0 \leq i, j \leq 1$.

Let $h : S^1 \rightarrow \mathbb{R}$ be a Morse function. We define a positive number K by

$$K := \inf_{x \in S^1} M(h)(x) > 0.$$

We define $g : S^1 \times S^1 \rightarrow \mathbb{R}$ by $g(x, y) := h(x) + h(y)$. Then $g_n(x_0, x_1, \dots, x_n) = h(x_0) + 2h(x_1) + \dots + 2h(x_{n-1}) + h(x_n)$. It is easy to see

$$\#\text{Cr}(g_n) = (\#\text{Cr}(h))^{n+1}.$$

The Hessian $\text{Hess}_x(g_n)$ is the diagonal matrix $\text{diag}(h''(x_0), 2h''(x_1), \dots, 2h''(x_{n-1}), h''(x_n))$. Hence

$$\inf_{u \in \mathbb{R}^{n+1}, \|u\|=1} \|\text{Hess}_x g_n(u)\| = \min(|h''(x_0)|, 2|h''(x_1)|, \dots, 2|h''(x_{n-1})|, |h''(x_n)|).$$

$$\begin{aligned} M(g_n)(x) &= \max(|h'(x_0)|, 2|h'(x_1)|, \dots, 2|h'(x_{n-1})|, |h'(x_n)|) \\ &\quad + \min(|h''(x_0)|, 2|h''(x_1)|, \dots, 2|h''(x_{n-1})|, |h''(x_n)|). \end{aligned}$$

This is bounded from below by

$$\min(|h'(x_0)| + |h''(x_0)|, 2|h'(x_1)| + 2|h''(x_1)|, \dots, \\ 2|h'(x_{n-1})| + 2|h''(x_{n-1})|, |h'(x_n)| + |h''(x_n)|) \geq K.$$

Therefore we get $M(g_n)(x) \geq K$ for all $n \geq 1$ and $x \in (S^1)^{n+1}$.

We define a open set $\mathcal{V} \subset C^\infty(S^1 \times S^1)$ as the set of $f \in C^\infty(S^1 \times S^1)$ satisfying $2\|\nabla(f - g)\|_\infty + 5\|\nabla^2(f - g)\|_\infty < K$. We will prove that for all $f \in \mathcal{V}$ and $n \geq 1$ the functions $f_n : (S^1)^{n+1} \rightarrow \mathbb{R}$ are Morse functions and

$$(11) \quad \#\text{Cr}(f_n) = \#\text{Cr}(g_n) = (\#\text{Cr}(h))^{n+1}.$$

Let $f \in \mathcal{V}$. By using (10)

$$\left| \frac{\partial f_n}{\partial x_i} \right| \geq \left| \frac{\partial g_n}{\partial x_i} \right| - 2\|\nabla(f - g)\|_\infty.$$

For any $u \in \mathbb{R}^{n+1}$ with $\|u\| = 1$,

$$\|\text{Hess}_x(f_n)u\| \geq \|\text{Hess}_x(g_n)u\| - 5\|\nabla^2(f - g)\|_\infty.$$

Therefore

$$M(f_n)(x) \geq M(g_n)(x) - 2\|\nabla(f - g)\|_\infty - 5\|\nabla^2(f - g)\|_\infty > K - K = 0.$$

Hence $M(f_n)(x) > 0$ for all $x \in (S^1)^{n+1}$. This shows that f_n is a Morse function.

For all $t \in [0, 1]$, the functions $tg + (1 - t)f : S^1 \times S^1 \rightarrow \mathbb{R}$ are contained in \mathcal{V} . Since non-degenerate critical points are persistent, this implies the equation (11).

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